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Journal of Geometry and Physics 29 (1999) 1–4

JOURNAL OF GEOMETRY AND PHYSICS

Brief Communication

Quantization of bounded domains

Andrea Loi [1]

University of Warwick, Coventry CV4 7AL, UK

Received 17 March 1998; received in revised form 29 June 1998

Abstract

We consider the quantization of a complex manifold endowed with the Bergman form following the ideas of Cahen, Gutt and Rawnsley. In particular we give a geometric interpretation for the quantization to be regular in terms of the Hilbert space of square integrable holomorphic n-forms on M and the Hilbert space of holomorphic n-forms on M bounded with respect to the Liouville element. © 1999 Published by Elsevier Science B.V. All rights reserved.

Subj. Class.: Quantum mechanics
1991 MSC: 53C55; 58F06
Keywords: Kähler metrics; Quantization

1. Preliminaries

A geometric quantization of a Kähler manifold (M, ω) is a pair (L, h), where L is a holomorphic line bundle over M and h is a hermitian structure on L such that $\mathrm{curv}(L, h) = -2\pi \mathrm{i}\omega$. The curvature $\mathrm{curv}(L, h)$ is calculated with respect to the *Chern connection.* If $\sigma : U \to L^+$ is a trivializing *holomorphic* section (where L^+ denotes the complement of the zero section in L) then the following formula holds:

$$\mathrm{curv}(L, h) = -\partial\bar{\partial} \log h(\sigma(x), \sigma(x)). \tag{1}$$

Let $(s_0, \ldots, s_N)$ $(N \leq \infty)$ be a unitary basis of the separable complex Hilbert space $\mathcal{H}_h$ of all global holomorphic sections s of L, which are bounded with respect to the Liouville element defined by

[1] E-mail: loi@vaxca1.unica.it

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PII S0393-0440(98)00035-7

$$\langle s, s\rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x)) \frac{\omega^n(x)}{n!}$$

(see [2–5]). Define the smooth function on M by

$$\epsilon_{(L,h)}(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{2}$$

Assuming that for all $x \in M$ there exists $s \in \mathcal{H}_h$ such that $s(x)$ is different from zero one can define the map

$$\phi_\sigma : U \to \mathbb{C}^{N+1} \setminus \{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_N(x)}{\sigma(x)}\right), \tag{3}$$

where $\sigma : U \to L^+$ is a trivializing holomorphic section on an open set $U \subset M$. It follows easily that (3) is the local expression of a globally defined map $\phi_{(L,h)} : M \to \mathbb{P}^N(\mathbb{C})$. This is called the *coherent states map*.

Theorem 1 [2]. *The coherent states map is full, i.e. $\phi(M)$ is not contained in $\mathbb{P}^r(\mathbb{C})$ with $r < N$. Furthermore*

$$\phi^*_{(L,h)} \Omega^N_{FS} = \omega + \frac{\mathrm{i}}{2\pi} \partial\bar{\partial} \log \epsilon_{(L,h)}. \tag{4}$$

Corollary 1. *Let (L, h) be a quantization of a Kähler manifold (M, ω). If $\epsilon_{(L,h)}$ is a positive constant, then ω is projectively induced.*

Recall that a Kähler form ω on a complex manifold M is *projectively induced* if there exists $N \leq \infty$ and a holomorphic map

$$\phi : M \to \mathbb{P}^N(\mathbb{C})$$

such that

$$\phi^* \Omega^N_{FS} = \omega. \tag{5}$$

In the proof of Theorem 3 we need

Theorem 2 (cf. [6]). *Let $N, N' \leq \infty$. Let M be a complex manifold, $\phi : M \to \mathbb{P}^N(\mathbb{C})$ and $\psi : M \to \mathbb{P}^{N'}(\mathbb{C})$ full holomorphic maps such that $\phi^* \Omega^N_{FS} = \psi^* \Omega^{N'}_{FS}$ is a Kähler form on M. Then $N = N'$ and there exists a unitary transformation U of $\mathbb{P}^N(\mathbb{C})$ such that $\phi = U \circ \psi$.*

2. The results

Let M be an n-dimensional complex manifold and K its canonical bundle. If α is a holomorphic section of K, i.e. a holomorphic n-form on M then in a complex coordinate system U, endowed with local coordinates $(z_1, \dots, z_n)$, there exists a holomorphic function f_α such that

$$\alpha(z) = f_\alpha(z)\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \quad \forall z \in U. \tag{6}$$

Let $(\alpha_0, \dots, \alpha_{N'})$ $(N' \leq \infty)$ be a unitary basis for the separable complex Hilbert space $(\mathcal{F}, (\cdot, \cdot))$ of all holomorphic n-forms α bounded with respect to $\|\alpha\|^2 = (\alpha, \alpha) := (\mathrm{i}^n/2^n) \int_M \alpha \wedge \bar{\alpha}$. Let K^* be the smooth function on U given by

$$K^*(z, \bar{z}) = \sum_{j=0}^{N'} f_{\alpha_j}(z) \bar{f}_{\alpha_j}(z). \tag{7}$$

The expression $\partial\bar{\partial} \log K^*$ does not depend on the coordinates. Hence $\omega_B = (\mathrm{i}/2\pi)\partial\bar{\partial} \log K^*$ is a globally defined two-form on M. In the sequel we will suppose ω_B is a Kähler form . For example this happens for the bounded domains in $\mathbb{C}^N$ (see [7] for details).

Formula

$$h(\alpha, \alpha) := \frac{|f_\alpha|^2}{K^*} \quad \forall \alpha \in H^0(K) \tag{8}$$

defines a hermitian structure on K and it is immediate to verify that the pair (K, h) is a geometric quantization for (M, ω_B).

Furthermore it follows from (2) that

$$\epsilon_{(K,h)} = \sum_{j=0}^{N} |f_{s_j}|^2 / K^*, \tag{9}$$

where $s_j = f_{s_j}\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, $j = 0, \dots, N$, is a unitary basis for $(\mathcal{H}_h, \langle \cdot , \cdot \rangle_h)$.

Theorem 3. *Let M be a complex manifold such that ω_B is a Kähler form. Then $\epsilon_{(K,h)}$ equals a positive constant λ if and only if the following conditions are satisified:*
(i) *the complex dimension of $\mathcal{F}$ is equal to the complex dimension of $\mathcal{H}_h$;*
(ii) $(\cdot, \cdot) = \lambda \langle \cdot , \cdot \rangle_h$.

Proof. Suppose that $\dim \mathcal{H}_h = \dim \mathcal{F} = N + 1$ and $\langle \cdot , \cdot \rangle_h = \lambda(\cdot, \cdot)$. Let α_j and s_j, $j = 0, 1, \dots N$, be unitary bases for $(\mathcal{F}, (\cdot , \cdot))$ and $(\mathcal{H}_h, \langle \cdot , \cdot \rangle_h)$, respectively. Then there exist an $(N + 1) \times (N + 1)$ unitary matrix u_{jk} and a complex number C such that

$$f_{s_j} = C \sum_{k=0}^{N} u_{jk} f_{\alpha_k}, \quad j = 0, \dots, N, \tag{10}$$

and $|C|^2 = \lambda$. Hence, by (9),

$$\epsilon_{(K,h)} = \sum_{j=0}^{N} |f_{s_j}|^2 / K^* = \sum_{j=0}^{N} |f_{s_j}|^2 \Big/ \sum_{j=0}^{N} |f_{\alpha_j}|^2 = \lambda.$$

Conversely, suppose that $\epsilon_{(K,h)} = \lambda$ and let $N+1$ be the dimension of $\mathcal{H}_h$. By Theorem 1 the coherent states map

$$\phi_{(L,h)} : M \to \mathbb{P}^N(\mathbb{C}) : x \mapsto [(f_{s_0}(x), \ldots, f_{s_N}(x))]$$

is a full holomorphic map and $\phi^*_{(L,h)}\Omega^N_{FS} = \omega_B$. On the other hand ω_B is projectively induced via the full holomorphic map

$$j : M \to \mathbb{P}^{N'}(\mathbb{C}) : x \mapsto [(f_{\alpha_0}(x), \ldots, f_{\alpha_{N'}}(x))].$$

(see [7] for details). By Theorem 2 $N = N'$, i.e. $\dim \mathcal{H}_h = \dim \mathcal{F} = N+1$. Moreover, there exist an $(N+1) \times (N+1)$ unitary matrix u_{jk} and a complex number C such that (10) holds, i.e. $(\cdot,\cdot) = |C|^2\langle \cdot\, , \,\cdot \rangle_h$. and, by the proof of the first part, $|C|^2 = \lambda$. □

Theorem 4. *Let M be a simply connected complex manifold such that ω_B is Kähler–Einstein with scalar curvature -1 and $\epsilon_{(K,h)}$ is a positive constant, say λ. Then $K^* = \lambda \det(g_{j\bar{k}})$ (see* [7, p.274] *for a discussion of this condition).*

Proof. Let $\omega_B = (\mathrm{i}/2)\sum_{j,\bar{k}=1}^{n} g_{j\bar{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_{\bar{k}}$ be the expression of the Bergman form in local coordinates $(z_1, \ldots, z_n)$. If ω_B is Kähler–Einstein then

$$\omega_B = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial} \log K^* = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial} \log \det(g_{j\bar{k}}) = -\rho_{\omega_B}, \tag{11}$$

where ρ_{ω_B} is the Ricci form on M (see [1])

This is equivalent to $\partial\bar{\partial} \log\{K^*/\det(g_{j\bar{k}})\} = 0$. It follows from the simply connectness of M that there exists a holomorphic function k on M such that $K^*/\det(g_{j\bar{k}}) = \mathrm{e}^{\Re(k)}$.

Since $\epsilon_{(K,h)}$ is constant if follows from Theorem 3 that $\dim \mathcal{H}_h = \dim \mathcal{F} = N+1$. Then it is not hard to see that $s_j := \mathrm{e}^{k/2}\alpha_j$ is a unitary basis for $(\mathcal{H}_h, \langle \cdot\, , \,\cdot \rangle_h)$, where $(\alpha_0, \ldots, \alpha_N)$ is a unitary basis for $(\mathcal{F}, (\cdot,\cdot))$. By formula (9) and by hypothesis $\epsilon_{(K,h)} = \mathrm{e}^{\Re(k)} = \lambda$. □

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